# On the Reduction of the Eigenvalue Problem for Spinning Axisymmetric Structures

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**This paper presents an entirely new method for the calculation of the natural frequencies and natural modes of a spinning axisymmetric spacecraft with flexible appendages. The approach is based on a method for the solution of the eigenvalue problem for general linear gyroscopic systems developed recently by this author. When the structure is axisymmetric, the matrices defining the eigenvalue problem are of a special type. The present paper takes advantage of the special nature of the system to reduce the order of the eigenvalue problem by a factor of two. Moreover, it develops a computational algorithm that solves the "reduced eigenvalue problem" more efficiently than existing algorithms. The method should find its application to a large variety of spinning axisymmetric structures.**

## Introduction

THE problem of spinning flexible structures has received a great deal of attention in recent years due to a multitude of space applications. One aspect of the problem of particular interest is the behavior of flexible systems when perturbed from steady rotation. For small perturbations, the problem falls in the general class of linear gyroscopic systems. Quite recently, the author of this paper has developed a modal analysis for linear gyroscopic systems,[1,2] thus permitting an entirely new approach to the response problem of spinning flexible spacecraft. The method involves the solution of the eigenvalue problem for complete rotating systems, where the problem is defined by one symmetric and one skew symmetric matrix. It is shown in Ref. 1 that the solution consists of pure imaginary eigenvalues, appearing in pairs of complex conjugates, and associated complex conjugate eigenvectors. Reference 1 also presents a method whereby the problem is reduced to a standard eigenvalue problem in terms of symmetric matrices alone, the solution of which is known to be real. Reference 2 develops a modal analysis that uses the solution of the eigenvalue problem of Ref. 1 to generate a closed-form response to arbitrary initial excitation and external excitation.

When the system possesses axial symmetry, the matrices defining the eigenvalue problem have special structures. Indeed, the elements of the matrices are so arranged and they possess such values that the matrices can be partitioned into $2 \times 2$ submatrices that can be operated with as if they were single elements. Introducing two special $2 \times 2$ matrices, where the first is the identity matrix and the second can be interpreted as the matrix counterpart of the imaginary number $i=\sqrt{-1}$, an algorithm permitting the reduction of the order of the eigenvalue problem by a factor of two is presented. However, whereas the full eigenvalue problem is in terms of a real symmetric matrix, the "reduced eigenvalue problem" is in terms of a Hermitian matrix of a special type, in the sense that its elements are either real or they are pure imaginary. The present paper develops a computational algorithm designed to take advantage of this special form of Hermitian matrices. The algorithm uses real algebra alone and is based on Jacobi's method for the diagonalization of real symmetric matrices;[3] it will be referred to as the modified Jacobi method. A solution of the reduced eigenvalue problem by the modified Jacobi method is considerably more efficient than a solution of the full eigenvalue problem by the ordinary Jacobi method. It should be pointed out that the reduction idea is by no means limited to the Jacobi method. Indeed, the reduction idea has been used by this author to produce a modified power method using matrix deflation, and the concept can be used to produce a modified Householder's method.

As an illustration, the method was applied to the solution of the eigenvalue problem of a spinning axisymmetric rigid body with flexible rods extending along the spin axis. Solutions were obtained for both the reduced eigenvalue problem and the full problem; the modified Jacobi method produced a solution more than four times as fast as the ordinary Jacobi method. This can be attributed to the facts that, for a desired accuracy, the modified Jacobi method requires half the number of iterations required by the ordinary Jacobi method and the number of nonzero elements to be calculated during each iteration step is also smaller by a factor of two.

The same system was investigated by this author in an earlier paper,[4] in which the system natural frequencies were obtained by solving the characteristic equation numerically. The natural frequencies obtained in the present paper are in good agreement with those obtained in Ref. 4. On the other hand, Ref. 4 made no attempt to calculate natural modes, as the significance of the concept was not recognized at that time.

## General Problem Formulation

Let us consider a structure spinning uniformly in space and examine the perturbed motion of the structure from the steady spin. The perturbed motion can be described in terms of the angular displacements of a suitable system of body axes and the elastic motion of the flexible parts relative to these axes. In general, the angular perturbations can be described by a maximum set of three coordinates depending on time alone, whereas the elastic motion can be described by an appropriate number of coordinates depending on space and time. The mathematical formulation consists of a set of ordinary differential equations for the rotational motion and a set of partial differential differential equations and appropriate boundary conditions for the elastic motion, where the two sets of equations are coupled. Such a system has come to be known as hybrid. We shall concern ourselves with the case in which the angular perturbations from steady rotation and the elastic displacements are small, so that the system of equations is linear.

Quite frequently it is more convenient to work with a discrete mathematical formulation instead of a hybrid for-

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mulation. A discrete formulation consists entirely of ordinary differential equations and can be obtained by a so-called "discretization" process. The most common discretization methods are the lumped parameter method, the finite element method, and the assumed-modes method, where the latter is often referred to as the Rayleigh-Ritz method. A detailed discussion of all three discretization methods can be found in Ref. 5.

Let us consider the perturbed motion of a discrete (or discretized) $n$-degree-of-freedom from steady rotation and assume that the motion can be described by $2n$ first-order linear ordinary differential equations having the matrix form

$$I\dot{x}(t)+Gx(t)=X(t) \tag{1}$$

where $I$ and $G$ are $2n\times 2n$ real nonsingular matrices, the first symmetric and the second skew symmetric, $x(t)$ is the $2n$--dimensional state vector, and $X(t)$ is the $2n$-dimensional vector of nonconservative forces. Note that the elements of the state vector $x(t)$ consist of the angular perturbations of a given frame from uniform rotation and the elastic displacements relative to the frame as well as the corresponding angular and elastic velocities.

The eigenvalue problem corresponding to system (1) has the form

$$\lambda Ix+Gx=0 \tag{2}$$

It was shown in Ref. 1 that the solution of Eq. (2) consists of $2n$ eigenvalues in the form of the pure imaginary complex conjugates $\lambda_r=\pm i\omega_r\ (r=1,2,\ldots,n)$ and $n$ pairs of associated eigenvectors $x_r=y_r+iz_r$ and $\bar{x}_r=y_r-iz_r$, where $y_r$ and $z_r$ are the real and the imaginary parts of the eigenvectors. It was further shown in Ref. 1 that the complex eigenvalue problem (2) can be transformed into the real eigenvalue problem

$$\omega_r^2 Iy_r=Ky_r,\quad \omega_r^2 Iz_r=Kz_r,\quad r=1,2,\ldots,n \tag{3}$$

where

$$K=G^T I^{-1} G \tag{4}$$

is a real symmetric matrix. Hence, the problem has been rendered into a *standard eigenvalue problem in terms of two real symmetric matrices* similar to that for nonrotating structures.

From Eqs. (3), we conclude that $\omega_r^2$ is a double eigenvalue to which there correspond two eigenvectors, namely, $y_r$ and $z_r$. Assuming that $I$ is positive definite, it is shown in Ref. 1 that $K$ is also positive definite, from which it follows that the eigenvectors $y_r$ and $z_r$ are all independent, and hence they are orthogonal. Moreover, if they are normalized so that $y_r^T Iy_r=z_r^T Iz_r=1\ (n=1,2,\ldots,n)$, then they satisfy the orthogonality relations

$$y_r^T Iy_s=z_r^T Iz_s=\delta_{rs};\quad y_r^T Iz_s=z_r^T Iy_s=0\quad r,s=1,2,\ldots,n \tag{5}$$

Because the eigenvectors $y_r$ and $z_r$ $(r=1,2,\ldots,n)$ constitute a set of $2n$ independent vectors, they form a basis in a $2n$-dimensional vector space, a fact used in Ref. 2 to develop the modal response for linear gyroscopic systems.

## Special Algorithm for Axially Symmetric Systems

When the system possesses axial symmetry one of the rotational coordinates is ignorable. Moreover, matrices $I$ and $G$ possess special structures. Indeed, the elements of $I$ and $G$ are so arranged and they have such values that the matrices $I$ and $G$ can be partitioned in a way that permits a reduction in the order of the eigenvalue problem by a factor of two. More specifically, if $I$ and $G$ are $2n\times 2n$ matrices, then they can be partitioned into $2\times 2$ submatrices that can be operated with as if they were single elements. The object of this paper is to develop a computational algorithm capable of taking advantage of the special structures of $I$ and $G$.

Let us assume that $n$ is odd, $n=2k+1$, and consider the case in which the matrix $I$ can be written in the convenient partitioned form

$$I=\left[\begin{array}{ccccccc}
(I_{1,1})1 & (I_{1,2})1 & ---- & (I_{1,k+1})i & (I_{1,k+2})i & ---- & (I_{1,2k+1})1 \\
(I_{1,2})1 & (I_{2,2})1 & ---- & (I_{2,k+1})i & (I_{2,k+2})i & ---- & (I_{2,2k+1})1 \\
\hline
-(I_{1,k+1})i & -(I_{2,k+1})i & ---- & (I_{k+1,k+1})1 & (I_{k+1,k+2})1 & ---- & -(I_{k+1,2k+1})i \\
-(I_{1,k+2})i & -(I_{2,k+2})i & ---- & (I_{k+1,k+2})1 & (I_{k+2,k+2})1 & ---- & -(I_{k+2,k+2})1 \\
\hline
(I_{1,2k+1})1 & (I_{2,2k+1})1 & ---- & (I_{k+1,2k+1})i & (I_{k+2,2k+1})i & ---- & (I_{2k+1,2k+1})1
\end{array}\right] \tag{6}$$

in which

$$1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad i=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{7}$$

where $1$ is recognized as the unit matrix of order two. On the other hand, the second matrix in Eq. (7) was denoted symbolically by $i$ because of the analogy with the imaginary number $i=\sqrt{-1}$. Indeed, we have

$$i^2=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=-1 \tag{8}$$

Note that, although the matrix $I$ is of the order $2n$, the above notation permits us to regard $I$ as being only of order $n$, with every element being multiplied by either $1$, $i$, or $-i$. In a similar manner, let us assume that the matrix $G$ can be written in the reduced form

$$G=\left[\begin{array}{ccccccc}
(G_{1,1})i & (G_{1,2})i & ---- & -(G_{1,k+1})1 & -(G_{1,k+2})1 & ---- & (G_{1,2k+1})i \\
(G_{1,2})i & (G_{2,2})i & ---- & -(G_{2,k+1})1 & -(G_{2,k+2})1 & ---- & (G_{2,2k+1})i \\
\hline
(G_{1,k+1})1 & (G_{2,k+1})1 & ---- & (G_{k+1,k+1})i & (G_{k+1,k+2})i & ---- & (G_{k+1,2k+1})1 \\
(G_{1,k+2})1 & (G_{2,k+2})1 & ---- & (G_{k+1,k+2})i & (G_{k+2,k+2})1 & ---- & (G_{k+2,2k+1})1 \\
\hline
(G_{1,2k+1})i & (G_{2,2k+1})i & ---- & -(G_{k+1,2k+1})1 & -(G_{k+2,2k+1})1 & ---- & (G_{2k+1,2k+1})i
\end{array}\right] \tag{9}$$

It can be verified that the matrix $K$, as given by Eq. (4) has precisely the same structure as the matrix $I$. Moreover, all the matrix operations required to generate $K$ can be carried out by operating with the partitioned matrices in $I$ and $G$ as if they were single elements. In this regard, it should be observed that $i^T = -i$.

For the matrix equations (3) to make mathematical sense, it is necessary to construct an $n$-dimensional vector representing $y$ or $z$. To this end, let us introduce the two-dimensional vector $\mathbf{1}$ whose elements are equal to unity, i.e.,

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{10}$$

It follows from definitions (7) that

$$1\,\mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad i\,\mathbf{1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{11}$$

In view of the above, a vector $v$ having the form

$$v = [v_1 v_1 v_2 v_2 \cdots - v_{k+1} v_{k+1} - v_{k+2} v_{k+2} \cdots v_{2k+1} v_{2k+1}]^T \tag{12}$$

can be written symbolically as

$$v = \left[\begin{array}{c} (v_1) 1\,\mathbf{1} \\ (v_2) 1\,\mathbf{1} \\ \hdashline (v_{k+1}) i\,\mathbf{1} \\ (v_{k+2}) i\,\mathbf{1} \\ \hdashline (v_{2k+1}) 1\,\mathbf{1} \end{array}\right] = \left[\begin{array}{c} (v_1) 1 \\ (v_2) 1 \\ \hdashline (v_{k+1}) i \\ (v_{k+2}) i \\ \hdashline (v_{2k+1}) 1 \end{array}\right] \mathbf{1} \tag{13}$$

where the vector enclosed by brackets on the extreme right of Eq. (13) can be regarded as an $n$-dimensional vector. With the understanding that multiplication on the right by $\mathbf{1}$ converts an $n$-dimensional vector into a $2n$-dimensional vector, we can operate with $v$ as an $n$-dimensional vector by simply ignoring $\mathbf{1}$ on the right side of Eq. (13). The full 2n-dimensional vector can be recovered at a later stage. In a similar manner, the $2n$-dimensional vector

$$v = [-v_1 v_1 - v_2 v_2 \ldots v_{k+1} v_{k+1} v_{k+2} v_{k+2} \cdots - v_{2k+1} v_{2k+1}]^T \tag{14}$$

can be written symbolically as the $n$-dimensional vector

$$v = [(v_1) i (v_2) i \ldots (v_{k+1}) 1 (v_{k+3}) 1 \ldots (v_{2k+1}) i]^T \tag{15}$$

and operated with as such, provided we recall that the right side of Eq. (15) is really multiplied on the right by $\mathbf{1}$. Note that the forms (13) and (15) of the vector $v$ were chosen so as to be compatible with the matrices $I$ and $K$, in the sense that multiplication of $v$ on the left by $I$ or by $K$ yields a vector similar in structure to $v$ itself. The products $Iv$ and $Kv$ can be more easily visualized if we regard the elements of the matrices $I$ and $K$ and those of the vectors $v$ as real numbers if they are multiplied by $1$ and as pure imaginary numbers if they are multiplied by $i$. In view of the above, we can write the eigenvalue problem (3) in the *reduced form*

$$\omega^2 I v = K v \tag{16}$$

where $I$ and $K$ are $n \times n$ matrices and $v$ is an $n$-dimensional vector and it represents either $y$ or $z$ or a linear combination of $y$ and $z$.

It remains to produce a computational algorithm capable of taking advantage of the special form of the reduced eigenvalue problem, Eq. (16), for the purpose of obtaining an efficient solution of the problem.

## Solution of the Eigenvalue Problem by a Modified Jacobi Method

There are many methods for the solution of the eigenvalue problem associated with real symmetric matrices, for which the eignvalues are known to be real. Some of the most familiar ones are the power method, Jacobi's method, Given's method, Householder's method, and the QR algorithm in conjunction with inverse iteration. All of these methods, or a combination thereof, can be used to solve the real eigenvalue problem of order $2n$, Eqs. (3). In reducing the order of the eigenvalue problem from $2n$ to $n$, however, the matrices $I$ and $K$ can no longer be regarded as being real and symmetric nor can the vector $v$ be regarded as being real. In fact, matrices $I$ and $K$ are Hermitian but of a special type since their elements are either real or pure imaginary. To treat the eigenvalue problem (16) as complex would defeat the purpose and, indeed, our goal is to devise a method capable of solving the reduced eigenvalue problem (16) while retaining all the computational advantages associated with real symmetric matrices. If such a method can be devised, then substantial computational savings can be realized by solving the reduced eigenvalue problem (16) insteady of the full problem (3). In the following we propose to develop a method for the solution of the reduced eigenvalue based on the Jacobi method. The method is appreciably faster than the ordinary Jacobi method. It should be pointed out, however, that the same general ideas can be applied to develop computational algorithms based on other methods, such as the Householder method.

Let us consider a real symmetric matrix $A$ and assume that is has the same structure as $I$ (or $K$). Next, let us write the eigenvalue problem associated with $A$ in the matrix form

$$AU = U\gamma \tag{17}$$

where $U$ is the modal matrix of $A$ and $\gamma$ is the diagonal matrix of the eigenvalues of $A$. Multiplying Eq. (17) on the left by $U^{-1}$, we obtain

$$U^{-1} A U = \gamma \tag{18}$$

so that the solution of the eigenvalue problem of $A$ can be obtained by simply diagonalizing $A$. It can be easily demonstrated that the modal matrix $U$ is orthonormal, i.e., it satisfies

$$U^{-1} = U^T \tag{19}$$

The diagonalization of a real symmetric matrix can be effected by a large variety of methods. Some of these methods can be used for Hermitian matrices. The matrix $A$ has a special structure, however, so that the interest lies in a method that retains all the computational advantages of a real symmetric matrix. To this end, we wish to explore a modification of the Jacobi method. Hence, let us introduce the "rotation matrix"

$$R = \begin{array}{c} \begin{array}{cccccccccc} \phantom{1} & \phantom{0} & & p & & & & q & & \end{array} \\ \left[\begin{array}{cccccccccc} 1 & 0 & ---- & 0 & ---- & 0 & ---- & 0 & ---- & 0 \\ 0 & 1 & ---- & 0 & ---- & 0 & ---- & 0 & ---- & 0 \\ 0 & 0 & ---- & (\cos\theta) 1 & ---- & 0 & ---- & -(\sin\theta) 1 & ---- & 0 \\ 0 & 0 & ---- & 0 & ---- & 1 & ---- & 0 & ---- & 0 \\ 0 & 0 & ---- & (\sin\theta) 1 & ---- & 0 & ---- & (\cos\theta) 1 & ---- & 0 \\ 0 & 0 & ---- & 0 & ---- & 0 & ---- & 0 & ---- & 1 \end{array}\right] \begin{array}{c} \\ \\ p \\ \\ q \\ \\ \end{array} \end{array} \tag{20}$$

and consider the case in which the element in the row $p$ and column $q$ of the matrix $A$ is "real", i.e., it has the form $(A_{pq})1$. Next, let us define the matrix $B$ as the triple matrix product

$$B = R^T A R \tag{21}$$

Then, it can be easily verified that the matrix $B$ has exactly the same structure as the structure of $A$ and its elements have the values

$$\begin{aligned}
B_{pp} &= A_{pp}\cos^2\theta + 2A_{pq}\sin\theta\cos\theta + A_{qq}\sin^2\theta \\
B_{qq} &= A_{pp}\sin^2\theta - 2A_{pq}\sin\theta\cos\theta + A_{qq}\cos^2\theta \\
B_{pq} &= (A_{qq} - A_{pp})\sin\theta\cos\theta + A_{pq}(\cos^2\theta - \sin^2\theta) \\
&\left.\begin{aligned} B_{pr} &= A_{pr}\cos\theta + A_{qr}\sin\theta \\ B_{qr} &= -A_{pr}\sin\theta + A_{qr}\cos\theta \end{aligned}\right\} \quad r \neq p,q \\
B_{rs} &= A_{rs}, \quad r,s \neq p,q
\end{aligned} \tag{22}$$

If in Eqs. (22) $r<p$, then $A_{pr}$ and $B_{pr}$ must be replaced by $A_{rp}$ and $B_{rp}$, respectively. Similarly, if $r<q$, then $A_{qr}$ and $B_{qr}$ must be replaced by $A_{rq}$ and $B_{rp}$, respectively. Note that, because of the structure of the matrices $A$ and $B$, it is only necessary to calculate the elements on the main diagonal and those above the main diagonal. In the case in which the element in row $p$ and column $q$ of the matrix $A$ is "pure imaginary", i.e., it has the form $(A_{pq})i$, we can use the rotation matrix

$$R^* = \begin{array}{cc}
\begin{array}{cccccccc} \phantom{1} & \phantom{0} & & p & & q & & \end{array} & \\
\left[\begin{array}{cccccccc}
1 & 0 & \cdots & 0 & \cdots 0 \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots 0 \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & (\cos\theta)1 & \cdots 0 \cdots & -(\sin\theta)i & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots 1 \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & (\sin\theta)i & \cdots 0 \cdots & (\cos\theta)1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots 0 \cdots & 0 & \cdots & 1
\end{array}\right] & \begin{array}{c} \\ \\ p \\ \\ q \\ \\ \end{array}
\end{array} \tag{23}$$

Indeed, the triple matrix product

$$B = R^{*T} A R^* \tag{24}$$

yields once again a matrix $B$ having the same structure as $A$ and whose elements are given by Eqs. (22). On the other hand, if the element in the row $p$ and column $q$ of $A$ has the form $-(A_{pq})i$, then the rotation matrix to be used is $R^{**} = R^{*T}$. It is easy to verify that also in the case the triple matrix product

$$B = R^{**T} A R^{**} \tag{25}$$

yields a matrix having the same structure as $A$ and whose elements are given by Eqs. (22).

The fact that all three matrix products, Eqs. (21), (24), and (25) yield the same matrix elements, Eqs. (22), forms the basis for a matrix diagonalization algorithm using successive rotations. The procedure, which for ordinary real symmetric matrices is known as the Jacobi method, consists of annihilating the element $B_{pq}$ at every iteration step by choosing the angle $\theta$ so as to satisfy

$$\tan 2\theta = 2A_{pq}/(A_{pp} - A_{qq}) \tag{26}$$

Then, it can be shown (see Ref. 3, Sec. 4-8) that the sum of the squares of the diagonal elements of $B$ increases at the expense of the sum of the squares of the off-diagonal elements. The result of the iteration process is the diagonal matrix $\gamma$ of the eigenvalues of $A$. Denoting by $R_\ell$ ($\ell = 1,2,\ldots,m$) the rotation matrices associated with the various iteration steps, regardless of whether they are of the type $R$, $R^*$, or $R^{**}$, we can write

$$\lim_{m\to\infty} R_m^T R_{m-1}^T \ldots R_2^T R_1^T A R_1 R_2 \ldots R_{m-1} R_m = \gamma \tag{27}$$

Moreover, because the matrices $R_\ell$ are orthonormal, we conclude from Eqs. (18), (19), and (27) that the modal matrix of $A$ is simply

$$U = \lim_{m\to\infty} R_1 R_2 \ldots R_{m-1} R_m \tag{28}$$

Of course, in practice a finite number of iterations suffices.

Next, let us return to the eigenvalue problem (16). Introducing the transformation

$$v' = I^{1/2} v \tag{29}$$

the eigenvalue problem (16) reduces to

$$\omega^2 v' = K' v' \tag{30}$$

where

$$K' = I^{-1/2} K I^{-1/2} = I^{-1/2} G^T I^{-1} G I^{-1/2} \tag{31}$$

is a matrix of the type (6). In fact, the eigenvalue problem (30) is of the type (17), so that it can be solved by the modified Jacobi method just developed. Note, however, that before the matrix $K'$ can be generated, it is necessary to solve the eigenvalue problem associated with $I$. This problem also can be solved by the procedure described above. The solution of the eigenvalue problem associated with $K'$ yields the actual eigenvalues $\omega_r^2$. On the other hand, to recover the acutal modal vectors, we must write

$$v_r = I^{-1/2} v_r' \tag{32}$$

Note that, because the eigenvectors $v_r'$ are orthonormal in an ordinary sense, the modal vectors $v_r$ are orthonormal with respect to the matrix $I$, so that the procedure yields automatically eigenvectors that satisfy Eqs. (5).

The reduced eigenvalue problem yields $n$ eigenvalues $\omega_r^2$ and $n$ eigenvectors $v_r$ ($r = 1,2,\ldots,n$). It is known, however, that the full eigenvalue problem possesses $n$ double eigenvalues and $n$ pairs of associated eigenvectors. The second eigenvector of the reduced eigenvalue problem corresponding to a given $\omega_r^2$ can be obtained by simply observing that if $v_r$ is a solution of Eq. (16), then $iv_r$ is also a solution, where $iv_r$ is a vector obtained by multiplying every element of $v_r$ by $i$.

Finally, it must be recalled that the elements of the $n$-dimensional eigenvectors $v_r$ and $iv_r$ are multiplied by either $1$ or $i$, where $1$ and $i$ are matrices and not ordinary numbers. To recover the full $n$-dimensional eigenvectors, we must multiply the elements of $v_r$ on the right by the vectors **1**, where **1** is defined by Eq. (10). It should be pointed out that, because $v_r$ and $iv_r$ are two eigenvectors corresponding to the same eigenvalue $\omega_r^2$, any linear combination of $v_r$ and $iv_r$ is also an eigenvector. This permits us to determine $y_r$ and $z_r$.

## Spinning Axisymmetric Body. Discretization of the Equations of Motion

Let us consider a symmetric rigid body with mass moments of inertia $A$, $B=A$, and $C$ about the body axes $x$, $y$, and $z$, respectively. Two symmetric flexible rods lie along axis $z$ when in equilibrium, where the equilibrium state is defined as uniform angular velocity $\Omega_z=\Omega=\text{const}$ about axis $z$ with the rods undeformed. If perturbed slightly, the body acquires small nutational velocities $\Omega_x \ll \Omega$ and $\Omega_y \ll \Omega$ as well as small elastic deformations $u_x(z,t)$ and $u_y(z,t)$, as shown in Fig. 1. Denoting by $\frac{1}{2}\rho(z)$ the mass density of the rods and by $\frac{1}{2}EI(z)$ their flexural rigidity, it is shown in Ref. 4 that the linearized equations of motion of the system are

$$\rho\ddot{u}_x+\frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2 u_x}{\partial z^2}\right]-\rho\Omega^2u_x-2\rho\Omega\dot{u}_y+\rho z(\Omega\Omega_x+\dot{\Omega}_y)=0$$

$$\rho\ddot{u}_y+\frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2 u_y}{\partial z^2}\right]-\rho\Omega^2u_y+2\rho\Omega\dot{u}_x+\rho z(\Omega\Omega_y-\dot{\Omega}_x)=0$$

$$A'\dot{\Omega}_x+\Omega(C-A')\Omega_x-\int_h^{h+\ell}\rho z(\ddot{u}_y+\Omega\dot{u}_x)\,dz -\Omega\int_h^{h+\ell}\rho z(\dot{u}_x-\Omega u_y)dz=0 \tag{33}$$

$$A'\dot{\Omega}_y-\Omega(C-A')\Omega_y+\int_h^{h+\ell}\rho z(\ddot{u}_x-\Omega\dot{u}_y)\,dz -\Omega\int_h^{h+\ell}\rho z(\dot{u}_y+\Omega u_x)dz=0$$

where $A'=A+\int_h^{h+\ell}\rho z^2dz$ is the moment of inertia of the entire body in undeformed state about a transverse axis. The displacements $u_x$ and $u_y$ are subject to the boundary conditions

$$u_x=\partial u_x/\partial z=0,\quad u_y=\partial u_y/\partial z=0\quad \text{at}\ x=h$$

$$\left.\begin{aligned}EI(\partial^2u_x/\partial z^2)=(\partial/\partial z)[EI(\partial^2u_x/\partial z^2)]=0\\ EI(\partial^2u_y/\partial z^2)=(\partial/\partial z)[EI(\partial^2u_y/\partial z^2)]=0\end{aligned}\right\}\ \text{at}\ x=h+\ell \tag{34}$$

Note that Eqs. (33) imply antisymmetric elastic motion, $u_x(-z,t)=-u_x(z,t)$ and $u_y(-z,t)=-u_y(z,t)$.

Introducing the notation

$$w_x=\dot{u}_x-\Omega u_y,\quad w_y=\dot{u}_y+\Omega u_x \tag{35}$$

where $w_x$ and $w_y$ can be interpreted as elastic velocities of the non-nutating body, Eq. (33) can be transformed into the set of first-order differential equations in time

$$\rho(\dot{u}_x-\Omega u_y-w_x)=0$$

$$\rho(\dot{u}_y-\Omega u_x-w_y)=0$$

$$\rho\dot{w}_x+\rho z\dot{\Omega}_y+\frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2u_x}{\partial z^2}\right]-\rho\Omega w_y+\rho\Omega z\Omega_x=0$$

$$\rho\dot{w}_y-\rho z\dot{\Omega}_x+\frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2u_y}{\partial z^2}\right]+\rho\Omega w_x+\rho\Omega z\Omega_y=0 \tag{36}$$

$$-\int_h^{h+\ell}\rho z\dot{w}_ydz+A'\dot{\Omega}_x-\Omega\int_h^{h+\ell}\rho zw_xdz+(C-A')\Omega\Omega_y=0$$

$$\int_h^{h+\ell}\rho z\dot{w}_xdz+A'\dot{\Omega}_y-\Omega\int_h^{h+\ell}\rho zw_ydz-(C-A')\Omega\Omega_x=0$$

where the equations are still subject to the boundary conditions (34).

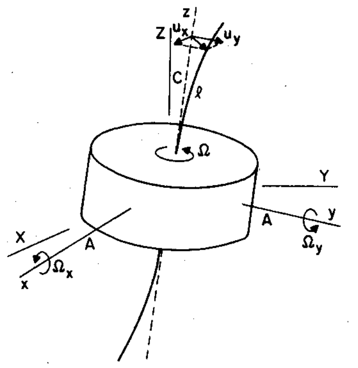


**Fig. 1 The spinning axisymmetric structure.**

Equations (36) (as well as Eqs. (33)) represent a set of hybrid differential equations of motion. We shall discretize the system by the so-called assumed-modes method[3] whereby the elastic displacements are assumed to have the form of the series

$$u_x(z,t)=\sum_{i=1}^{k}\phi_i(z)u_{xi}(t)$$

$$u_y(z,t)=\sum_{i=1}^{k}\phi_i(z)u_{yi}(t) \tag{37}$$

where $u_{xi}(t)$ and $u_{yi}(t)$ are generalized displacements and $\phi_i(z)$ are comparison functions[3] satisfying the eigenvalue problem defined by the differential equation

$$\frac{d^2}{dz^2}\left[EI\frac{d^2\phi_i}{dz^2}\right]=\rho\Lambda_i^2\phi_i,\quad i=1,2,\ldots \tag{38}$$

and the boundary conditions

$$\phi_i(h)=0,\quad \left.\frac{d\phi_i}{dz}\right|_{z=h}=0$$

$$i=1,2,\ldots \tag{39}$$

$$EI\left.\frac{d^2\phi_i}{dz^2}\right|_{z=h+\ell}=0,\quad \frac{d}{dz}\left[EI\frac{d^2\phi_i}{dz^2}\right]\Bigg|_{z=h+\ell}=0$$

In addition, the comparison functions are normalized so as to satisfy

$$\int_h^{h+\ell}\rho\phi_i\phi_jdz=\delta_{ij},\quad i,j=1,2,\ldots \tag{40a}$$

where $\delta_{ij}$ is the Kronecker delta. It follows immediately that

$$\int_h^{h+\ell}\frac{d^2}{dz^2}\left[EI\frac{d^2\phi_i}{dz^2}\right]\phi_jdz=\Lambda_i^2\delta_{ij},\quad i,j=1,2,\ldots \tag{40b}$$

Note that the comparison functions $\phi_i$ are merely the eigenfunctions corresponding to the fixed-base, nonrotating rods. Inserting series (37) into Eqs. (35), we obtain

$$w_x(z,t)=\sum_{i=1}^{k}\phi_i(z)w_{xi}(t)$$

$$w_y(z,t)=\sum_{i=1}^{k}\phi_i(z)w_{yi}(t) \tag{41}$$

where

$$w_{xi}(t)=\dot{u}_{xi}(t)-\Omega u_{yi}(t)$$

$$w_{yi}(t)=\dot{u}_{yi}(t)+\Omega u_{xi}(t) \tag{42}$$

play the role of generalized elastic velocities. Finally, introducing Eqs. (37) and (41) into Eqs. (36), multiplying the first four of Eqs. (36) by $\phi_j(z)$, integrating the resulting equations over the elastic domain, and using Eqs. (40), we obtain the discrete set of $4k+2$ equations

$$\left.\begin{aligned}
&\Lambda_i^2(\dot{u}_{xi}-\Omega u_{yi}-w_{xi})=0\\
&\Lambda_i^2(\dot{u}_{yi}+\Omega u_{xi}-w_{yi})=0\\
&\dot{w}_{xi}+a_i\dot{\Omega}_y+\Lambda_i^2u_{xi}-\Omega w_{yi}+\Omega a_i\Omega\hat{x}=0\\
&\dot{w}_{yi}-a_i\dot{\Omega}_x+\Lambda_i^2u_{yi}+\Omega w_{xi}+\Omega a_i\Omega\hat{y}=0
\end{aligned}\right\}\quad i=1,2,\dots,k$$

$$-\sum_{j=1}^{k} a_j\dot{w}_{yj}+A'\dot{\Omega}_x-\Omega\sum_{j=1}^{k} a_jw_{xj}+(C-A')\Omega\Omega_y=0 \tag{43}$$

$$\sum_{j=1}^{k} a_j\dot{w}_{xj}+A'\dot{\Omega}_y-\Omega\sum_{j=1}^{k} a_jw_{yj}-(C-A')\Omega\Omega_x=0$$

where the notation

$$a_i=\int_{\ell}^{h+\ell}\rho z\phi_i\,\mathrm{d}z,\quad i=1,2,\dots,k \tag{44}$$

has been introduced. Note that the first $2k$ of Eqs. (43) were multiplied by $\Lambda_i^2$ to render the full set of equations in the desired form. It should be pointed out that the assumed-modes method did not uncouple the system but merely discretized it.

Equstions (43) are precisely of the form (1). Moreover, very importantly, the problem is of the type that lends itself to the reduction scheme. Indeed, recalling the matrices $1$ and $i$, Eqs. (7), matrices $I$ and $G$ can be written in the reduced form

$$I=\left[\begin{array}{ccc|ccc|c}
(\Lambda_1^2)1 & 0 & \cdots & & & & \\
0 & (\Lambda_2^2)1 & \cdots & & 0 & & 0\\
\hline
 & & & 1 & 0 & \cdots & -(a_1)i\\
 & 0 & & 0 & 1 & \cdots & -(a_2)i\\
\hline
 & 0 & & (a_1)i & (a_2)i & \cdots & (A')1
\end{array}\right] \tag{45}$$

and

$$G=\left[\begin{array}{ccc|ccc|c}
(\Omega\Lambda_1^2)i & 0 & \cdots & -(\Lambda_1^2)1 & 0 & \cdots & \\
0 & (\Omega\Lambda_2^2)i & \cdots & 0 & -(\Lambda_2^2)1 & \cdots & 0\\
\hline
(\Lambda_1^2)1 & 0 & \cdots & (\Omega)i & 0 & \cdots & (\Omega a_1)1\\
0 & (\Lambda_2^2)1 & \cdots & 0 & (\Omega)i & \cdots & (\Omega a_2)1\\
\hline
 & 0 & & -(\Omega a_1)1 & -(\Omega a_2)1 & \cdots & [-(C-A')\Omega]i
\end{array}\right] \tag{46}$$

In addition, introducing the notation

$$u_{xyi}=\begin{bmatrix}u_{xi}\\u_{yi}\end{bmatrix},\quad w_{xyi}=\begin{bmatrix}w_{xi}\\w_{yi}\end{bmatrix},\quad \Omega_{xy}=\begin{bmatrix}\Omega_x\\\Omega_y\end{bmatrix} \tag{47}$$

$$x=[u_{xy1}^Tu_{xy2}^T\dots w_{xy1}^Tw_{xy2}^T\dots\Omega_{xy}^T]^T \tag{48}$$

Before we can carry out the diagonalization of the matrix $I$, the generation of the matrix $K'$, and the diagonalization of $K'$, it is necessary to assign numerical values to the system parameters.

## Numerical Solution of the Reduced Eigenvalue Problem

Let us consider the case in which the rots are uniform, $EI(z)=EI=\text{const}$ and $\rho(z)=\rho=\text{const}$, and the system parameters have the values:

$$\ell/h=0.5,\quad \Lambda_1=60\text{ rad s}^{-1},\quad \Omega=30\text{ rad s}^{-1} \tag{49}$$

$$A=1.0\text{ lb-ft-s}^{-2},\quad A'=1.1\text{ lb-ft-s}^{-2},\quad C=1.5\text{ lb-ft-s}^{-2}$$

From Ref. 3 (Sec. 5-10), we conclude that if $\Lambda_1$ is specified, then all remaining $\Lambda_i=(i=2,3,\dots)$ are uniquely determined. The functions $\phi_i(z)$ can also be obtained from Ref. 3 except for their amplitudes; these are determined uniquely by Eqs. (40a). Then recognizing that $\int_h^{h+\ell}\rho z^2\,dz=0.1$, we can use Eqs. (44) and obtain the values $a_i\,(i=1,2,\dots)$. This permits us to evaluate the matrices $I$ and $G$.

To generate the matrix $K'$, it is necessary to solve first the eigenvalue problem associated with the matrix $I$. Before this can be accomplished, however, we must specify the number $k$ of terms we wish to retain in series (37). We propose to investigate two cases, namely, $k=1$ and $k=2$. The investigation of these two cases should permit an assessment of the series truncation effect, at least for this particular system. Hence, let us generate the matrix $K'$ for the case $k=2$, with the observation that for the case $k=1$ we must merely omit from $K'$ the second and fourth rows and columns. Solving the eigenvalue problem associated with $I$ by the modified Jacobi method, calculating $I^{-1/2}$, and using Eq. (31), we obtain

$$K' = \begin{bmatrix} (4{,}756)1 & (722)1 & (3{,}601)i & (1)i & (229)1 \\ (722)1 & (144{,}327)1 & (4)i & (22{,}563)i & (648)1 \\ -(3{,}601)i & -(4)i & (4{,}701)1 & (1{,}734)1 & -(-495)i \\ -(1)i & -(22{,}563)i & (1{,}734)1 & (143{,}809)1 & -(-8{,}386)i \\ (229)1 & (648)1 & (-495)i & (-8{,}386)i & (780)1 \end{bmatrix} \tag{50}$$

The solution of the eigenvalue problem associated with $K'$ yields the eigenvalues $\omega_r^2$ and the eigenvectors $v_r'$ and $iv_r'$. Then, using Eq. (32), we obtain the actual modal vectors $v_r$ and $iv_r$, which can be used to determine $y_r$ and $z_r$.

Solving the case $k=1$, which amounts to solving the eigenvalue problem associated with the matrix obtained by removing the second and fourth rows and columns from (50), and performing the vector transformations indicated above, we obtain the natural frequencies and the modal vectors

$$\omega_1 = 9.035067 \text{ rad s}^{-1}$$

$$y_1 = \begin{bmatrix} 0 \\ -0.004363 \\ 0.170324 \\ 0 \\ 0 \\ 0.865180 \end{bmatrix} \quad z_1 = \begin{bmatrix} 0.004363 \\ 0 \\ 0 \\ 0.170324 \\ -0.865180 \\ 0 \end{bmatrix} \tag{51a}$$

$$\omega_2 = 35.949634 \text{ rad s}^{-1}$$

$$y_2 = \begin{bmatrix} -0.010895 \\ 0 \\ 0 \\ -0.718534 \\ -0.461494 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} 0 \\ -0.010895 \\ 0.718534 \\ 0 \\ 0 \\ -0.461494 \end{bmatrix} \tag{51b}$$

$$\omega_3 = 91.212125 \text{ rad s}^{-1}$$

$$y_3 = \begin{bmatrix} 0.011834 \\ 0 \\ 0 \\ -0.724348 \\ -0.105883 \\ 0 \end{bmatrix} \quad z_3 = \begin{bmatrix} 0 \\ 0.011834 \\ 0.724358 \\ 0 \\ 0 \\ -0.105883 \end{bmatrix} \tag{51c}$$

On the other hand, for $k=2$, i.e., using the full matrix (50), we obtain

$$\omega_1 = 9.030410 \text{ rad s}^{-1}$$

$$y_1 = \begin{bmatrix} 0 \\ -0.004362 \\ 0 \\ -0.000029 \\ 0.170231 \\ 0 \\ 0.001140 \\ 0 \\ 0 \\ 0.865066 \end{bmatrix} \quad z_1 = \begin{bmatrix} 0.004362 \\ 0 \\ 0.000029 \\ 0 \\ 0 \\ 0.170231 \\ 0 \\ 0.001140 \\ -0.865066 \\ 0 \end{bmatrix} \tag{52a}$$

$$\omega_2 = 35.94311 \text{ rad s}^{-1}$$

$$y_2 = \begin{bmatrix} -0.010896 \\ 0 \\ 0.000027 \\ 0 \\ 0 \\ -0.718494 \\ 0 \\ 0.001770 \\ -0.461030 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} 0 \\ -0.010896 \\ 0 \\ 0.000027 \\ 0.718494 \\ 0 \\ -0.001770 \\ 0 \\ 0 \\ -0.461030 \end{bmatrix} \tag{52b}$$

$$\omega_3 = 91.211840 \text{ rad s}^{-1}$$

$$y_3 = \begin{bmatrix} 0.011834 \\ 0 \\ -0.000006 \\ 0 \\ 0 \\ -0.724351 \\ 0 \\ 0.000349 \\ -0.105858 \\ 0 \end{bmatrix} \quad z_3 = \begin{bmatrix} 0 \\ 0.011834 \\ 0 \\ -0.000006 \\ 0.724351 \\ 0 \\ -0.000349 \\ 0 \\ 0 \\ -0.105858 \end{bmatrix} \tag{52c}$$

$$\omega_4 = 349.081901 \text{ rad s}^{-1}$$

$$y_4 = \begin{bmatrix} 0 \\ 0.000069 \\ 0 \\ 0.001880 \\ -0.026130 \\ 0 \\ -0.712483 \\ 0 \\ 0 \\ 0.094947 \end{bmatrix} \quad z_4 = \begin{bmatrix} -0.000069 \\ 0 \\ -0.001880 \\ 0 \\ 0 \\ -0.026130 \\ 0 \\ -0.712483 \\ -0.094947 \\ 0 \end{bmatrix} \tag{52d}$$

$$\omega_5 = 408.439039 \text{ rad s}^{-1}$$

$$y_5 = \begin{bmatrix} 0 \\ 0.000055 \\ 0 \\ 0.001881 \\ 0.020693 \\ 0 \\ 0.711900 \\ 0 \\ 0 \\ -0.075200 \end{bmatrix} \quad z_5 = \begin{bmatrix} -0.000055 \\ 0 \\ -0.001881 \\ 0 \\ 0 \\ 0.020693 \\ 0 \\ 0.711900 \\ 0.075200 \\ 0 \end{bmatrix} \tag{52e}$$

The above solutions were obtained by the modified Jacobi method.

Comparing Eqs. (51) and (52), we note that there is not much difference between the first three natural frequencies calculated by using one term in series (37) and those obtained by using two terms. This is in line with conclusions reached in Ref. 4, namely, that the truncation effect is not significant for the problem at hand. Consistent with this, we observe that there is not much difference between the natural modes associated with the first three natural frequencies, which is to be expected. Note, however, that for every additional term in series (37), two additional higher natural frequencies and natural modes are obtained. The nature of the response problem may require the knowledge of higher frequencies.

For comparison, the full eigenvalue problem was solved by the ordinary Jacobi method. The solution of the reduced eigenvalue problem was more than four times faster than the solution of the full problem. This can be attributed to the facts that: 1) for the same desired accuracy, the number of iterations required for the solution of the eigenvalue problem by the modified Jacobi method is one half the number required by the ordinary Jacobi method and 2) the number of nonzero elements to be calculated per iteration step is also smaller by a factor of two. Another way of explaining the efficiency of the modified Jacobi method compared to the ordinary Jacobi method is that the reduced eigenvalue problem eliminates the duplication involved in performing identical computations twice, as is the case with the full eigenvalue problem.

## Summary and Conclusions

This paper presents a method for the solution of the eigenvalue problem for spinning axisymmetric structures that is considerably more efficient than existing ones. The method takes advantage of the special nature of the matrices defining the eigenvalue problem to develop an algorithm that reduces the order of the problem by a factor of two. Then, a computational algorithm capable of solving the reduced eigenvalue problem is presented. The algorithm is based on the Jacobi method for the diagonalization of symmetric matrices and should be at least four times as efficient as the Jacobi method. Using the ideas presented in this paper, other computational algorithms capable of solving efficiently the reduced eigenvalue problem, such as a modified Householder's method, can be developed.

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